Int J Theor Phys (2007) 46: 2836–2842
DOI 10.1007/s10773-007-9395-1

A Novel Moment Approach for Calculation of the Perron–Frobenius Spectrum

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Published online: 26 April 2007

Abstract The spectral properties of the Perron–Frobenius operator of the one-dimensional maps are studied by using the moment. In this paper we make an investigation into the properties of self-similar measures related to the theory of orthogonal polynomials. Numerical investigation of a particular family of maps shows that the spectrum generates the invariant measure. Analytical considerations generalize the results to a broader class of the maps. Some examples of this method are presented through out the paper.

Keywords Invariant measure · Orthogonal polynomials · Stieltjes transform · Chaotic dynamical system

1 Introduction

One of the most powerful ways of describing the dynamical features of systems, specially those having a very complicated geometrical and topological structure of individual orbits, is through invariant probability measures. Any such measure can be decomposed into ergodic components and, by a simple application of Birkhoffs Ergodic Theorem, almost every initial condition in each ergodic component has the same statistical distribution in space [1–3].

The important property of these invariant measures is, roughly speaking, that they lend weight to a region in phase space based on the degree of the probability by which "typical" trajectories may cross this region. The so-called Sinai–Ruelle–Bowen (SRB) measures, had already been studied in dynamical systems theory [3, 4]. The SRB-measure fully characterizes the transport properties, such as the transport law, transport coefficients and their

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fluctuations. Invariant measures or the stationary densities [5] provide a useful way to study the asymptotic behavior of dynamical systems. All these SRB measures are, in a sense, one-dimensional: they are absolutely continuous with respect to Lebesgue measure. The main idea of the approach is to define an operator, the Perron–Frobenius operator [6, 7], in the space of probability measures whose fixed points are invariant measures. There exist various methods such as perturbation theory [8, 9], maximum entropy approach [10], Galerkin method [11], and etc., for computing the Perron–Frobenius spectrum. The direct study of the spectrum of the Perron–Frobenius operator involves a series of problems, related to both the suitable definition of the Perron–Frobenius operator and relevant function space in which the eigenvalue problem is to be solved [12]. Generalized versions of the Perron–Frobenius operator have long been discussed in mathematics [13, 14]. In the physical literature, they appeared first in connection with the transient chaotic dynamics of one-dimensional maps and were then extended to study permanent chaotic behavior.

In this paper, by using the orthogonal polynomials we introduce methods for the calculation of the invariant measure. The introduced method based on the calculation of moments will be used for the given discrete-time dynamical system ($x_{n+1} = \Phi(x_n, \alpha)$). This algorithm can be used in discrete-time dynamical system with zero odd moments. By using some simple examples, we will show its applications which in turn will demonstrate the robustness of our new approach. The paper is organized as follows: in Sect. 2 we present the basic concepts of polynomials. The introduced algorithms are described in Sect. 3 and in Sect. 4 some application of the method on discrete time dynamical systems are presented. This section is followed by an outlook section.

2 Orthogonal Polynomials

Polynomials have already been discussed in different theories such as rook theory [15], matching theory [16] and etc. Throughout this paper, we focus only on those examples related to the calculation of the invariant measure. In this section, we determine the structure of a thin, irreducible module for a subconstituent algebra in a P- and Q-polynomial scheme [17, 18] to do so, let Π be the space of polynomials and μ a measure with an infinite number of increasing points. Then, there exists a unique sequence of normalized orthogonal polynomials $\{\hat{P}(x)\}_{n=1}^{\infty}$ associated with it (see [2] or [8, 20, 26]).

$\{P_n(x)\}$ is called an orthogonal polynomial sequence with respect to the weight function $\mu(x)$ on (a,b). Also a sequence $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial sequence which is proportional to a moment functional, L, provided that the two following condition for all nonnegative integers m and n, are met.

(a) $L[P_m(x)P_n(x)] = 0,\ m \neq n,\ n = 0, 1, 2, \ldots.$
(b) $L[P_n^2(x)] \neq 0.$

In this paper “Orthogonal Polynomial Sequence” will be abbreviated as “OPS” which may include both $\{P_n(x)\}$ and $\{Q_n(x)\}$.

3 Calculation Invariant Measure Using Stieltjes Transform Approach

There exist various methods to find invariant measures. One approach is to use the so-called Perron–Frobenius operator. When this operator is applied to $\mu(x)$, the density at nth time step will result in the density at $(n+1)$th the time step.

Let us recall that for a deterministic map $\Phi(x)$, the invariant probability measure $\mu(x)$ is the eigenfunction of the Perron–Frobenius (PF) operator L related to maximum eigenvalue 1 [7, 19].

$$L\mu(x) = \mu(x), \tag{1}$$

where the operator L is defined as [20]:

$$L\Phi(x) = \int \delta(y - f(x))\Phi(y)dy = \sum_{z=f^{-1}(x)} \frac{\Phi(z)}{|f'(z)|}. \tag{2}$$

Now by considering $d\mu(y) = \delta(y - f(x))d\nu(x)$ the density at the nth time becomes:

$$y^n d\mu(y) = \delta(y - f(x))y^n d\nu(x). \tag{3}$$

By integrating (3), we have:

$$\int_a^b y^n d\mu(y) = \int_a^b \delta(y - f(x))y^n d\nu(x) = \int_a^b (f(x))^n d\nu(x) \tag{4}$$

that can be presented in the following form either:

$$\mu_n = \int_a^b x^n d\mu = \int_a^b (f(x))^n d\mu. \tag{5}$$

The following integral exist in any integrable function:

$$L[f] = \int_a^b f(x)d\mu(x) \tag{6}$$

by using (5), equation (6) is reduced:

$$L[x^n] = \mu_n, \quad n = 0, 1, 2, \dots$$

where L is linear. Now, the calculation of Perron–Frobenius spectrum should be done in the following steps:

(1) In the first step, by using (5), we calculate the moment for the given discrete-time dynamical systems ($x_{m+1} = \Phi(x_m, \alpha)$).
(2) In the second step, we have to take L as our moment functional with moment sequence $\{\mu_n\}$. A necessary and sufficient condition for the existence of an OPS for L is (see, [21], Theorem 3.1):

$$\Delta_n \neq 0, \quad n = 0, 1, 2, 3, \dots.$$

Where Δ_n is defined as the following Hankel determinants:

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{vmatrix}.$$

(3) One of the most important characteristics of orthogonal polynomials is the fact that any three consecutive polynomials are connected by a very simple relation. Due to this fact, in the third step we have to take L as our quasi-definite moment functional and $\{P_n(x)\}$ as the corresponding monic OPS. A polynomial $\{P_n(x)\}$ is monic if the coefficient of x^n therein is 1. There exist constants c_n and $\lambda_n \neq 0$ which result in (see, [21], Theorem 3.2):

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}, \quad n = 2, 3, \ldots, \tag{7}$$

where $P_0(x) = 1$, $P_1(x) = x$ and $c_n = 0$ for all n.

(4) By considering any polynomial $P_n(x)$ of degree n, the following condition should be met (see [21], Theorems 3.2, 3.3):

$$L[\Pi_n(x)P_n(x)] = a_n L[x^n P_n(x)] = \frac{a_n k_n \Delta_n}{\Delta_{n-1}}, \quad \Delta_{-1} = -1,$$

where a_n denotes the leading coefficient of $\Pi_n(x)$ and k_n denotes the leading coefficient of $P_n(x)$. In the 4th step, we have calculated the λ_{n+1} in relation (7) (see, [21], Theorems 4.1 and 4.2):

$$\lambda_{n+1} = \frac{L[P_n^2(x)]}{L[P_{n-1}^2(x)]} = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2} \tag{8}$$

where $\Delta_{-1} = 1$ and $\Delta_0 = \mu_0 = \lambda_1$.

(5) If such a spectral distribution is unique, the spectral distribution μ is determined by the identity introduced with $G_\mu(x)$ in the 5th step:

$$G_\mu(x) = \cfrac{1}{x - \cfrac{\omega_1}{x - \cfrac{\omega_2}{x - \cfrac{\omega_3}{x - \cdots}}}} = \frac{P_{n-1}(x)}{P_n(x)} = \sum_{l=1}^{n} \frac{A_l}{x - x_l}, \tag{9}$$

where ω_n defined as follows (see, [21], Theorem 4.1):

$$\omega_n = \lambda_{n+1}, \quad n = 1, 2, \ldots$$

and $G_\mu(x)$ is called the Stieltjes transform and A_l is the coefficient in the Gauss quadrature formula based on the roots x_l of polynomial $P_n(x)$ [22].

(6) In the 6th step, we take $\{P_n(x)\}$ as the monic OPS of quasi-definite moment functional L, then (see, [21], Theorem 3.4):
(a) L is symmetric.
(b) $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$.
(c) In the corresponding recurrence formula (7), $c_n = 0$ ($n \geq 1$).
Turning to the recurrence (7), we observe that $(-1)^n P_n(x) = Q_n(x)$ meets:

$$Q_n(x) = (x + c)Q_{n-1}(x) - \lambda_n Q_{n-2}(x), \quad n \geq 2. \tag{10}$$

Therefore, if $Q_n(x) = P_n(x)$, then subtracting the above equation from (7) yields $2c_n P_{n-1}(x) = 0$, whence $c_n = 0$ for $n \geq 1$.

Conversely, if $c_n = 0$ in (7) for $n \geq 1$, then $\{Q_n(x)\}$ meet the same recurrence formula as $\{P_n(x)\}$. Since $Q_0(x) = P_0(x)$ and $Q_1(x) = P_1(x)$, we would have the equality, $Q_n(x) = P_n(x)$ for all n. (see, [21], Theorem 4.3).

(7) The coefficients A_l appearing in (9) are the same as Gauss quadrature constants and are calculated as follows:

$$A_l = \lim_{x \to x_l} (x - x_l) G_\mu(x), \tag{11}$$

where, x_l are the roots of polynomial $P_n(x)$.

The spectral distribution can be determined in the last step in terms of x_l, based on the following equation (for more details see Refs. [23–25]):

$$\mu = \sum_l A_l \delta(x - x_l). \tag{12}$$

4 Examples

In order to simplify our introduced model for determining the Perron–Frobenius spectrum, through two example, we have calculated the invariant measure.

- Discrete spectrum of Tchebichef map:

According to the introduced steps in Sect. 3, and in order to calculate the invariant measure for the Tchebichef map, we calculate the moments:

$$\mu_n = \int_0^1 \left(2x^2 - 1\right)^n d\mu = \sum_{k=0}^{n} 2^n \mu_{2n} (-1)^{n-k} C_k^n, \quad n = 0, 1, 2, \dots, \ k = 0, 1, \dots, n \tag{13}$$

where $C_k^n = \frac{n!}{k!(n-k)!}$ and $\mu_{2n} = \int_0^1 x^{2n} d\mu(x)$. It should be mentioned that $\mu_n = 0$ for all odd value of n. As an example by considering first five moments:

$$\mu_0 = 1, \qquad \mu_2 = 1/2, \qquad \mu_4 = 1/4. \tag{14}$$

At the second step, we calculate the coefficients λ_n according to (3.8):

$$\lambda_2 = \frac{1}{2}, \qquad \lambda_{3,4,5} = \frac{1}{4}. \tag{15}$$

In order to determine the $G_\mu(x)$, by considering the first two moments, we calculate both ω_2 and roots of P_2 ($P_2(x) = x^2 - 1/2$) then, we have:

$$G_\mu(x) = \frac{1}{x - \frac{\omega_1}{x}} = \frac{x}{x^2 - \frac{1}{2}} = \frac{x}{(x - \frac{\sqrt{2}}{2})(x + \frac{\sqrt{2}}{2})}. \tag{16}$$

In the last step with regard to the relation (11), Gauss quadrature constants are shaped as follows:

$$A_{l,2} = \lim_{x \to \pm\frac{\sqrt{2}}{2}} \left(x \pm \frac{\sqrt{2}}{2}\right) \frac{x}{(x - \frac{\sqrt{2}}{2})(x + \frac{\sqrt{2}}{2})} = \frac{1}{2}.$$

Now according to the definition of the invariant measure, we can write:

$$\mu = \frac{1}{2}\left(\delta\left(x - \frac{\sqrt{2}}{2}\right) + \delta\left(x + \frac{\sqrt{2}}{2}\right)\right).$$

By increasing the degree of the P polynomials, one could generate the realistic measure corresponding to the dynamical systems, such as shown by following steps. By increasing number of moments and increasing P_2 to P_3 the invariant measure can be written as:

$$\mu = \frac{1}{3}\left(\delta(x) + \delta\left(x - \frac{\sqrt{3}}{2}\right) + \delta\left(x + \frac{\sqrt{3}}{2}\right)\right)$$

and by considering P_4 the invariant measure would be written as follows:

$$\mu = \frac{1}{4}\left(\delta\left(x - \frac{\sqrt{2+\sqrt{2}}}{2}\right) + \delta\left(x - \frac{\sqrt{2-\sqrt{2}}}{2}\right) + \delta\left(x + \frac{\sqrt{2+\sqrt{2}}}{2}\right) + \delta\left(x + \frac{\sqrt{2-\sqrt{2}}}{2}\right)\right)$$

of course, the more you can increase the amount of n, the closer on get the reality. However due to avoid lengthening the present paper, we have only increased amount by 4.

- Tchebichef map in continues spectrum:

By choosing Tchebichef polynomials of first kind with scaling factor $\frac{1}{2}$ as orthogonal polynomials appearing in recurrence relation (7), i.e., $Q_n(x) = 2T_n(x/2)$, one can obtain parameters $\omega_1 = 2$, $\omega_k = 1$, $k = 2, 3, \ldots$, and $c_k = 0$, $k = 1, 2, 3, \ldots$. Thus $G_\mu(x)$ for this map can be written as:

$$G_\mu(x) = \frac{1}{n}\frac{T'_n(\frac{x}{2})}{T_n(\frac{x}{2})}. \tag{17}$$

Therefore, its spectral distribution can be shown as:

$$\mu = \frac{1}{n}\sum_{l=0}\delta\left(x - 2\cos\frac{(2l+1)\pi}{2n}\right), \quad l = 1, 2, \ldots \tag{18}$$

where $\cos\frac{(2l+1)\pi}{2n}$ are roots of the Tchebishef polynomial. Now, in the limit of large n, the relation (18) is reduced to continuous spectral distribution of $\mu(x)$:

$$\begin{aligned}\mu(x) &= \frac{1}{\pi}\int_0^\pi \delta(x - 2\cos(y))dy = \frac{1}{\pi}\int_0^\pi \frac{\delta(y - \arccos(x/2))}{2\sin(y)}dy \\ &= \frac{1}{\pi}\int_0^\pi \frac{\delta(y - \arccos(x/2))}{2\sin(y)}dy = \frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}, \quad -2 \le x \le 2\end{aligned} \tag{19}$$

which is the same as the continuous spectral distribution of Tchebishef polynomial.

5 Summary and Outlook

The invariant density $\mu(x)$ yields the distribution of points on the attractor generated by the map. In this paper we studied the properties of self-similar measures related to the theory of orthogonal polynomials. We showed several experiments from different applications demonstrating the robustness of our new approach. Experiments, presented in this paper, serve only the primary results of our designed approach. We need to perform more experiments in order to broaden our understanding of the presented approach. Zero odd moments have been assumed in our presented approach. As a further study, one can border the approach for discrete-time dynamical system by assuming non-zero odd moments.

Acknowledgements The authors express their sincere thanks to S. Salimi for very helpful discussions. They are also very grateful to Professor M.A. Jafarizadeh for his warm advice and encouragement.

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